Braided equivariant crossed modules and cohomology of Γ-modules

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Abstract

If Γ is a group, then braided Γ-crossed modules are classified by braided strict Γ-graded categorial groups. The Schreier theory obtained for Γ-module extensions of the type of an abelian Γ-crossed module is a generalization of the theory of Γ-module extensions.

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1 Introduction

Crossed modules have been used widely, and in various contexts, since their definition by Whitehead [21] in his investigation of the algebraic structure of second relative homotopy groups. Brown and Spencer [4] showed that crossed modules are determined by $\mathcal{G}$*-groupoids* (or *strict categorial groups*), and hence crossed modules can be studied by the theory of category. Thereafter, Joyal and Street [14] extended the result in [4] for *braided* crossed modules and *braided* strict categorial groups. A *braided strict categorial group* is a braided categorical group in which the unit, associativity constraints are strict and every object is invertible ($x \otimes y = 1 = y \otimes x$).

A brief summary of researches related to crossed modules was given in [6] by Carrasco et al. Results on the category of *abelian* crossed modules appeared in this work. Previously, the notion of abelian crossed module was characterized by that of the *center* of a crossed module in the paper of Norrie [16].

In [12], Fröhlich and Wall introduced the notion of graded categorical group. Thereafter, Cegarra and Khmaladze constructed the abelian (symmetric) cohomology of Γ-modules which was applied on the classification for braided (symmetric) Γ-graded categorical groups in [9] ([8]).

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The purpose of this paper is to study kinds of crossed modules which are defined by braided strict Γ-graded categorical groups. This result is an extension of the result of Joyal and Street mentioned above. After this introductory Section 1, Section 2 is devoted to recalling some necessary fundamental notions and results of braided (symmetric) graded categorical groups and factor sets of braided graded categorical groups. In Section 3 we show that the category $\mathbf{BrGr}^*$ of braided strict categorical groups and regular symmetric monoidal functors is equivalent to the category **BrCross** of braided crossed modules (Theorem 3.6). Each morphism in the category **BrCross** consists of a homomorphism $(f_1, f_0) : \mathcal{M} \to \mathcal{M}'$ of braided crossed modules and an element of the group of abelian 2-cocycles $Z^2_{ab}(\pi_0\mathcal{M}, \pi_1\mathcal{M}')$ in the sense of [11]. This result is a continuation of the result in [14] (Remark 3.1). It is obtained as a consequence of Classification Theorem 4.10.

In Section 4 we extend the result in Section 3 to graded structures by introducing the notions of *braided* Γ*-crossed module* and *braided strict* Γ*-graded categorical group* to classify braided Γ-crossed modules (see [15]). Theorem 4.10 states that the category ${}_{\mathbf{\Gamma}}\mathbf{BrGr}^*$ of braided strict Γ-graded categorical groups and regular Γ-graded symmetric monoidal functors is equivalent to the category ${}_{\mathbf{\Gamma}}\mathbf{BrCross}$ of braided Γ-crossed modules. Each morphism in the category ${}_{\mathbf{\Gamma}}\mathbf{BrCross}$ consists of a homomorphism $(f_1, f_0) : \mathcal{M} \to \mathcal{M}'$ of braided Γ-crossed modules and an element of the group of symmetric 2-cocycles $Z^2_{\Gamma,s}(\pi_0\mathcal{M}, \pi_1\mathcal{M}')$ in the sense of [9].

The problem of group extensions of the type of a crossed module has been mentioned in [19, 10, 3]. In Section 5 we show a treatment of the similar problem for Γ-module extensions of the type of an abelian Γ-crossed module. The Schreier theory for such extensions (Theorem 5.3) is presented by means of graded symmetric monoidal functors, and therefore we obtain the classification theorem of Γ-module extensions of the type of an abelian Γ-crossed module (Theorem 5.4).

The case of (non-braided) Γ-cossed modules is studied by Quang and Cuc in [17]. The results generalizes both the theory of group extensions of the type of a crossed module and the one of equivariant group extensions.

2 Preliminaries

2.1 Braided (symmetric) graded categorical groups

Let Γ be a fixed group, which we regard as a category with exactly one object, say $*$, where the morphisms are the members of Γ and the composition law is the group operation. A *grading* on a category $\mathbb{G}$ is then a functor, say $gr : \mathbb{G} \to \Gamma$. For any morphism u in $\mathbb{G}$ with $gr(u) = \sigma$, we refer to σ as the *grade* of u. The grading gr is said to be *stable* if for any $X \in \mathrm{Ob}\mathbb{G}$ and any $\sigma \in \Gamma$ there exists an isomorphism u in $\mathbb{G}$ with domain X such that $gr(u) = \sigma$.

A *braided* Γ*-graded monoidal category* [9] $\mathbb{G} = (\mathbb{G}, gr, \otimes, I, \mathbf{a}, \mathbf{r}, \mathbf{l}, \mathbf{c})$ consists of a category $\mathbb{G}$, a stable grading $gr : \mathbb{G} \to \Gamma$, graded functors $\otimes : \mathbb{G} \times_\Gamma \mathbb{G} \to \mathbb{G}$ and $I : \Gamma \to \mathbb{G}$, and graded natural equivalences defined by isomorphisms of grade 1 $\mathbf{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \overset{\sim}{\to} X \otimes (Y \otimes Z), \mathbf{l}_X : I \otimes X \overset{\sim}{\to} X, \mathbf{r}_X : X \otimes I \overset{\sim}{\to} X$ and $\mathbf{c}_{X,Y} : X \otimes Y \overset{\sim}{\to} Y \otimes X$ satisfying the following coherence conditions:

$$\mathbf{a}_{X,Y,Z\otimes T}\mathbf{a}_{X\otimes Y,Z,T} = (id_X \otimes \mathbf{a}_{Y,Z,T})\mathbf{a}_{X,Y\otimes Z,T}(\mathbf{a}_{X,Y,Z} \otimes id_T),$$

$$(id_X \otimes \mathbf{l}_Y)\mathbf{a}_{X,I,Y} = \mathbf{r}_X \otimes id_Y,$$

$$(id_Y \otimes \mathbf{c}_{X,Z})\mathbf{a}_{Y,X,Z}(\mathbf{c}_{X,Y} \otimes id_Z) = \mathbf{a}_{Y,Z,X}\mathbf{c}_{X,Y\otimes Z}\mathbf{a}_{X,Y,Z}, \tag{1}$$

$$(\mathbf{c}_{X,Z} \otimes id_Y)\mathbf{a}^{-1}_{X,Z,Y}(id_X \otimes \mathbf{c}_{Y,Z}) = \mathbf{a}^{-1}_{Z,X,Y}\mathbf{c}_{X\otimes Y,Z}\mathbf{a}^{-1}_{X,Y,Z}. \tag{2}$$

A *braided* Γ*-graded categorical group* [9] is a braided Γ-graded monoidal groupoid such that, for any object X, there is an object X' with an arrow $X \otimes X' \to 1$ of grade 1. If the braiding $\mathbf{c}$ is a symmetric constraint, that is, it satisfies the condition $\mathbf{c}_{Y,X} \circ \mathbf{c}_{X,Y} = id_{X\otimes Y}$ (in this case the relation (2) coincides with the relation (1)), then $\mathbb{G}$ is called a *symmetric* Γ-graded categorical group or a *graded Picard category* [8]. Then the subcategory $\operatorname{Ker}\mathbb{G}$ (whose objects are the objects of $\mathbb{G}$ and morphisms are the morphisms of grade 1 in $\mathbb{G}$) is a braided categorical group (a Picard category, respectively).

Let $(\mathbb{G}, gr)$ and $(\mathbb{G}', gr')$ be two (braided symmetric) Γ-graded categorical groups. A *graded symmetric monoidal functor* from $(\mathbb{G}, gr)$ to $(\mathbb{G}', gr')$ is a triple $(F, \widetilde{F}, F_*)$, where $F : (\mathbb{G}, gr) \to (\mathbb{G}', gr')$ is a Γ-graded functor, $\widetilde{F}_{X,Y} : FX \otimes FY \to F(X \otimes Y)$ are natural isomorphisms of grade 1 and $F_* : I' \to FI$ is an isomorphism of grade 1, such that the following coherence conditions hold:

$$\widetilde{F}_{X,Y\otimes Z}(id_{FX} \otimes \widetilde{F}_{Y,Z})\mathbf{a}_{FX,FY,FZ} = F(\mathbf{a}_{X,Y,Z})\widetilde{F}_{X\otimes Y,Z}(\widetilde{F}_{X,Y} \otimes id_{FZ}),$$

$$F(\mathbf{r}_X)\widetilde{F}_{X,I}(id_{FX} \otimes F_*) = \mathbf{r}_{FX},\ F(\mathbf{l}_X)\widetilde{F}_{I,X}(F_* \otimes id_{FX}) = \mathbf{l}_{FX},$$

$$\widetilde{F}_{Y,X}\mathbf{c}_{FX,FY} = F(\mathbf{c}_{X,Y})\widetilde{F}_{X,Y}.$$

Let $(F, \widetilde{F}, F_*), (F', \widetilde{F}', F'_*)$ be two graded symmetric monoidal functors. A *graded symmetric monoidal natural equivalence* $\theta : F \overset{\sim}{\to} F'$ is a graded natural equivalence such that, for all objects X, Y of $\mathbb{G}$, the following coherence conditions hold

$$\widetilde{F}'_{X,Y}(\theta_X \otimes \theta_Y) = \theta_{X\otimes Y}\widetilde{F}_{X,Y},\ \theta_I F_* = F'_*, \tag{3}$$

that is, a monoidal natural equivalence.

2.2 Braided (symmetric) graded categorical groups of type (M, N) and the theory of obstructions

Let $\mathbb{G}$ be a braided Γ-graded categorical group. We write $M = \pi_0(\mathrm{Ker}\,\mathbb{G}) = \pi_0\mathbb{G}$ for the abelian group of 1-isomorphism classes of the objects in $\mathbb{G}$ and $N = \pi_1(\mathrm{Ker}\,\mathbb{G}) = \pi_1\mathbb{G}$ for the abelian group of 1-automorphisms of the unit object of $\mathbb{G}$. Then $\mathbb{G}$ induces Γ-module structures on M, N and a normalized 3-cycocle $h \in Z^3_{\Gamma,ab}(M, N)$ in the sense of [9]. From these data, the authors of [9] constructed a braided Γ-graded categorical group, denoted by $\mathbb{G}(h)$ (or $\int_\Gamma(M, N, h)$), which is equivalent to $\mathbb{G}$. Below, we briefly recall this construction.

The objects of $\mathbb{G}(h)$ are the elements $s \in M$ and their arrows are pairs $(a, \sigma) : r \to s$ consisting of an element $a \in N$ and an element $\sigma \in \Gamma$ with $\sigma r = s$.

The composition of two morphisms $(r \xrightarrow{(a,\sigma)} s \xrightarrow{(b,\tau)} t)$ is defined by

$$(b, \tau) \circ (a, \sigma) = (b + \tau a + h(r, \tau, \sigma), \tau\sigma).$$

The graded tensor product is defined by

$$(r \overset{(a,\sigma)}{\to} s) \otimes (r' \overset{(b,\sigma)}{\to} s') = (rr' \xrightarrow{(a+sb+h(r,r',\sigma),\sigma)} ss').$$

The unit constraints are strict, $\mathbf{l}_s = (0, 1) = \mathbf{r}_s : s \to s$. The associativity and braiding constraints are, respectively, given by

$$\mathbf{a}_{r,s,t} = (h(r, s, t), 1) : (rs)t \to r(st),$$

$$\mathbf{c}_{r,s} = (h(r, s), 1) : rs \to sr.$$

The stable Γ-grading is defined by $gr(a, \sigma) = \sigma$. The unit graded functor $I : \Gamma \to \mathbb{G}(h)$ is defined by

$$I(* \xrightarrow{\sigma} *) = (1 \xrightarrow{(0,\sigma)} 1).$$

We call $\mathbb{G}(h)$ a *reduced* braided Γ-graded categorical group of $\mathbb{G}$. In the case when $\mathbb{G}$ is a Γ-graded Picard category, then $h \in Z^3_{\Gamma,s}$ in the sense of [8] and $\mathbb{G}(h)$ is a Γ-graded Picard category.

Let $\mathbb{G}$, $\mathbb{G}'$ be Γ-graded Picard categories, and let $\mathbb{G}(h) = \int_\Gamma(M, N, h)$, $\mathbb{G}'(h') = \int_\Gamma(M', N', h')$ be their reduced Γ-graded Picard categories, respectively. A graded functor $F : \mathbb{G}(h) \to \mathbb{G}'(h')$ is said to be *of type* (φ, f) if

$$F(s) = \varphi(s), \quad F(a, \sigma) = (f(a), \sigma), \quad s \in M, \ a \in N, \ \sigma \in \Gamma,$$

where $\varphi : M \to M'$; $f : N \to N'$ are homomorphisms of Γ-modules. Then the function

$$k = \varphi^* h' - f_* h$$

is called an *obstruction* of the functor F.

Based on the results on monoidal functors of type (φ, f) presented in [18], we obtain the following results with some appropriate modifications.

Proposition 2.1. *Let $\mathbb{G}, \mathbb{G}'$ be braided Γ-graded categorical groups, and let $\mathbb{G}(h)$, $\mathbb{G}'(h')$ be their reduced braided Γ-graded categorical groups, respectively.*

i) *Any graded symmetric monoidal functor $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{G}'$ induces a graded symmetric monoidal functor $\mathbb{G}(h) \to \mathbb{G}'(h')$ of type (φ, f).*

ii) *Any graded symmetric monoidal functor $\mathbb{G}(h) \to \mathbb{G}'(h')$ is a graded functor of type (φ, f).*

Proposition 2.2 ([8], Theorem 3.9)**.** *The graded functor $(F, \widetilde{F}) : \mathbb{G}(h) \to \mathbb{G}'(h')$ of type (φ, f) is realizable, that is, there are isomorphisms $\widetilde{F}_{x,y}$ so that $(F, \widetilde{F})$ is a graded symmetric monoidal functor, if and only if its obstruction $\overline{k}$ vanishes in $H^3_{\Gamma,s}(M, N')$. Then, there is a bijection*

$$\mathrm{Hom}_{(\varphi,f)}[\mathbb{G}(h), \mathbb{G}'(h')] \leftrightarrow H^2_{\Gamma,s}(M, N'),$$

where $\mathrm{Hom}_{(\varphi,f)}[\mathbb{G}(h), \mathbb{G}'(h')]$ denotes the set of homotopy classes of graded symmetric monoidal functors of type (φ, f) from $\mathbb{G}(h)$ to $\mathbb{G}'(h')$.

Note that $H^2_{\Gamma,s}(M, N') = H^2_{\Gamma,ab}(M, N')$.

2.3 Factor sets in braided graded categorical groups

According to the definition of a factor set with coefficients in a monoidal category [7], we now establish the following terminology.

Definition 2.3. A *symmetric factor set $\mathcal{F}$* on Γ with coefficients in a braided categorical group $\mathbb{G}$ (or a pseudofunctor from Γ to the category of braided categorical groups in the sense of Grothendieck [13]) consists of a family of symmetric monoidal auto-equivalences $F^\sigma : \mathbb{G} \to \mathbb{G}, \sigma \in \Gamma$, and isomorphisms between symmetric monoidal functors $\theta^{\sigma,\tau} : F^\sigma F^\tau \to F^{\sigma\tau}$, $\sigma, \tau \in \Gamma$ satisfying the conditions:

i) $F^1 = id_{\mathbb{G}}$,

ii) $\theta^{1,\sigma} = id_{F^\sigma} = \theta^{\sigma,1}$, $\sigma \in \Gamma$,

iii) for all $\sigma, \tau, \gamma \in \Gamma$, the following diagram commutes

$$\begin{array}{ccc} F^\sigma F^\tau F^\gamma & \xrightarrow{\theta^{\sigma,\tau} F^\gamma} & F^{\sigma\tau} F^\gamma \\ {\scriptstyle F^\sigma\theta^{\tau,\gamma}}\downarrow & & \downarrow{\scriptstyle \theta^{\sigma\tau,\gamma}} \\ F^\sigma F^{\tau\gamma} & \xrightarrow{\theta^{\sigma,\tau\gamma}} & F^{\sigma\tau\gamma}. \end{array}$$

We write $\mathcal{F} = (\mathbb{G}, F^\sigma, \theta^{\sigma,\tau})$, or simply (F, θ).

The following lemma comes from an analogous result on graded monoidal categories [7] or a part of Theorem 1.2 [20]. We sketch the proof since we need some of its details.

Lemma 2.4. *Any braided Γ-graded categorical group $(\mathbb{G}, gr)$ determines a symmetric factor set $\mathcal{F}$ on Γ with coefficients in a braided categorical group* $\mathrm{Ker}\,\mathbb{G}$.

Proof. For each $\sigma \in \Gamma$, we construct a symmetric monoidal functor $F^\sigma = (F^\sigma, \widetilde{F^\sigma}) : \mathrm{Ker}\,\mathbb{G} \to \mathrm{Ker}\,\mathbb{G}$ as follows. For any $X \in \mathrm{Ker}\,\mathbb{G}$, since the grading gr is stable, there is an isomorphism $\Upsilon^\sigma_X : X \xrightarrow{\sim} F^\sigma X$, where $F^\sigma X \in \mathrm{Ker}\,\mathbb{G}$, and $gr(\Upsilon^\sigma_X) = \sigma$. In particular, if $\sigma = 1$ we set $F^1 X = X$ and $\Upsilon^1_X = id_X$. For any morphism $f : X \to Y$ of grade 1 in $\mathrm{Ker}\,\mathbb{G}$, a morphism $F^\sigma(f)$ in $\mathrm{Ker}\,\mathbb{G}$ is determined by

$$F^\sigma(f) = \Upsilon^\sigma_Y \circ f \circ (\Upsilon^\sigma_X)^{-1}.$$

Natural isomorphisms $\widetilde{F}^\sigma_{X,Y} : F^\sigma X \otimes F^\sigma Y \xrightarrow{\sim} F^\sigma(X \otimes Y)$ are determined by

$$\widetilde{F}^\sigma_{X,Y} = (\Upsilon^\sigma_X \otimes \Upsilon^\sigma_Y) \circ (\Upsilon^\sigma_{X\otimes Y})^{-1}.$$

Moreover, for any pair $\sigma, \tau \in \Gamma$, there is an isomorphism between monoidal functors $\theta^{\sigma,\tau} : F^\sigma F^\tau \xrightarrow{\sim} F^{\sigma\tau}$, where $\theta^{1,\sigma} = id_{F^\sigma} = \theta^{\sigma,1}$, which is determined by

$$\theta^{\sigma,\tau}_X = \Upsilon^\sigma_{F^\tau X} \circ \Upsilon^\tau_X \circ (\Upsilon^{\sigma\tau}_X)^{-1},$$

for all $X \in \mathrm{Ob}\,\mathbb{G}$.

The pair (F, θ) determined above is a symmetric factor set. □

3 Braided crossed modules

We first recall that a *crossed module* [21] (B, D, d, ϑ) consists of groups B, D, group homomorphisms $d : B \to D,\ \vartheta : D \to \mathrm{Aut}B$ satisfying

C_1. $\vartheta d = \mu$,

C_2. $d(\vartheta_x(b)) = \mu_x(d(b)),\ x \in D, b \in B$,

where μ_x is an inner automorphism given by conjugation with x.

In this paper, the crossed module (B, D, d, ϑ) is sometimes denoted by $B \xrightarrow{d} D$, or by $d : B \to D$. For convenience, we write the addition for the operation in B and the multiplication for that in D.

The notion of braided crossed module over a groupoid was originally introduced by Brown and Gilbert in [2]. Later, the notion of braided crossed module over groups appeared in the work of Joyal and Street [14] (Remark 3.1).

Definition 3.1 ([14]). A *braided crossed module* $\mathcal{M}$ is a crossed module (B, D, d, ϑ) together with a map $\eta : D \times D \to B$ satisfying the following conditions:

C_3. $\eta(x, yz) = \eta(x, y) + \vartheta_y \eta(x, z)$,

C_4. $\eta(xy, z) = \vartheta_x \eta(y, z) + \eta(x, z)$,

C_5. $d\eta(x, y) = xyx^{-1}y^{-1}$,

C_6. $\eta(d(b), x) + \vartheta_x b = b$,

C_7. $\eta(x, d(b)) + b = \vartheta_x b$,

where $b \in B$, $x, y, z \in D$.

A braided crossed module is called a *symmetric* crossed module (see Aldrovandi and Noohi [1]) if $\eta(x,y)+\eta(y,x)=0$ for all $x,y\in D$. In this case, the conditions C_3 and C_4 coincide, the conditions C_6 and C_7 coincide.

The following properties follow from the definition of a braided crossed module.

Proposition 3.2. *Let $\mathcal{M}$ be a braided crossed module.*

i) $\eta(x,1)=\eta(1,y)=0$.

ii) $\operatorname{Ker} d$ *is a subgroup of* $Z(B)$.

iii) $\operatorname{Coker} d$ *is an abelian group.*

iv) *The homomorphism ϑ induces the identity on* $\operatorname{Ker} d$*, and hence the action of* $\operatorname{Coker} d$ *on* $\operatorname{Ker} d$*, given by*

$$sa=\vartheta_x(a),\quad a\in \operatorname{Ker} d,\ x\in s\in \operatorname{Coker} d,$$

is trivial.

The abelian groups $\operatorname{Ker} d$ and $\operatorname{Coker} d$ are also denoted by $\pi_1\mathcal{M}$ and $\pi_0\mathcal{M}$, respectively.

Example 3.3. *Let N be a normal subgroup of a group G so that the quotient group G/N is abelian, in other words, let N be a normal subgroup in G which contains the derived group (or the commutator subgroup) of G. Then, $(N,G,i,\mu,[,])$ is a braided crossed module, where $i:N\to G$ is an inclusion, $\mu:G\to \operatorname{Aut} N$ is defined by conjugation and $\eta:G\times G\to N$, $\eta(x,y)=[x,y](=xyx^{-1}y^{-1})$.*

According to Joyal and Street [14], each braided crossed module is determined by a braided strict categorical group. We now classify the category of braided crossed modules.

Definition 3.4. A *homomorphism* of braided crossed modules (B,D,d,ϑ,η) and $(B',D',d',\vartheta',\eta')$ consists of group homomorphisms $f_1:B\to B'$, $f_0:D\to D'$ such that:

H_1. $f_0d=d'f_1$,

H_2. $f_1(\vartheta_x b)=\vartheta'_{f_0(x)}f_1(b)$,

H_3. $f_1(\eta(x,y))=\eta'(f_0(x),f_0(y))$,

for all $x,y\in D$, $b\in B$.

Therefore, a homomorphism of braided crossed modules is that of crossed modules which satisfies H_3.

We determine the category

BrCross

whose objects are braided crossed modules and whose morphisms are triples (f_1,f_0,φ), where $(f_1,f_0):(B\xrightarrow{d}D)\to(B'\xrightarrow{d'}D')$ is a homomorphism

of braided crossed modules and $\varphi \in Z^2_{ab}(\mathrm{Coker}\, d, \mathrm{Ker}\, d')$. The composition with the morphism $(f'_1, f'_0, \varphi') : (B' \overset{d'}{\to} D') \to (B'' \overset{d''}{\to} D'')$ is given by

$$(f'_1, f'_0, \varphi') \circ (f_1, f_0, \varphi) = (f'_1 f_1, f'_0 f_0, (f'_1)_* \varphi + (f_0)^* \varphi'). \tag{4}$$

Definition 3.5. A symmetric monoidal functor $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{G}'$ is termed *regular* if

B_1. $F(x) \otimes F(y) = F(x \otimes y)$,
B_2. $F(b) \otimes F(c) = F(b \otimes c)$,
B_3. $\widetilde{F}_{x,y} = \widetilde{F}_{y,x}$,

for $x, y \in \mathrm{Ob}\mathbb{G}$, $b, c \in \mathrm{Mor}\mathbb{G}$.

Denote by

$$\mathbf{BrGr}^*$$

the category of braided strict categorical groups and regular symmetric monoidal functors and denote by $p : D \to \mathrm{Coker}\, d$ a canonical projection, we obtain the following classification result.

Theorem 3.6 (Classification Theorem)**.** *There exists an equivalence*

$$\begin{array}{rcl} \Phi : \mathbf{BrCross} & \to & \mathbf{BrGr}^*, \\ B \to D & \mapsto & \mathbb{G}_{B \to D} \\ (f_1, f_0, \varphi) & \mapsto & (F, \widetilde{F}) \end{array}$$

where $F(x) = f_0(x), F(b) = f_1(b), \widetilde{F}_{x,y} = \varphi(px, py)$, *for* $x, y \in D, b \in B$.

Proof. The proof of this theorem is a particular case of Theorem 4.10 in the next section. □

Remark 3.7. *Denote by* Br**Cross** *the subcategory of* **BrCross** *whose morphisms are homomorphisms of braided crossed modules* ($\varphi = 0$) *and denote by* Br**Gr*** *the subcategory of* **BrGr*** *whose morphisms are strict monoidal functors* ($\widetilde{F} = id$)*. Then, these two categories are equivalent via* Φ*.*

4 Braided Γ-crossed modules

The main objective of this section is to classify braided Γ-crossed modules by means of strict braided graded categorical groups. First, observe that if B is a Γ-group, the group $\mathrm{Aut}\, B$ of all automorphisms of B is also a Γ-group under the action

$$(\sigma f)(b) = \sigma(f(\sigma^{-1} b)),\ b \in B,\ f \in \mathrm{Aut}\, B,\ \sigma \in \Gamma.$$

Then, the map $\mu : B \to \mathrm{Aut}\, B, b \mapsto \mu_b$ (μ_b is an inner automorphism of B given by conjugation with b) is a homomorphism of Γ-groups.

Definition 4.1. Let B and D be Γ-groups. A *braided (symmetric)* Γ-*crossed module,* is a braided (symmetric) crossed module $\mathcal{M} = (B, D, d, \vartheta, \eta)$ in which $d : B \to D,\ \vartheta : D \to \mathrm{Aut}B$ are Γ-group homomorphisms satisfying the following conditions:

Γ_1. $\sigma(\vartheta_x(b)) = \vartheta_{\sigma x}(\sigma b)$,

Γ_2. $\sigma\eta(x, y) = \eta(\sigma x, \sigma y)$,

where $\sigma \in \Gamma,\ x, y \in D$ and $b \in B$.

Braided (symmetric) Γ-crossed modules are also called braided (symmetric) equivariant crossed modules by Noohi [15].

The following properties are implied from the definition of a braided Γ-crossed module.

Proposition 4.2. *Let $\mathcal{M}$ be a braided Γ-crossed module.*

i) $\mathrm{Ker}\, d$ *is a Γ-submodule of $Z(B)$.*

ii) $\mathrm{Coker}\, d$ *is a Γ-module under the action*

$$\sigma s = [\sigma x],\ x \in s \in \mathrm{Coker}\, d,\ \sigma \in \Gamma.$$

Example 4.3. *In Example 3.3, if G and N are Γ-groups, then $(N, G, i, \mu, [,])$ is a braided Γ-crossed module.*

Example 4.4. *Let $d : B \to D$ be a morphism of Γ-module and let D act trivially on B. Let $\eta : D \times D \to Kerd$ be a biadditive function satisfies Γ_2 and*

$$\eta|_{Imd \times D} = 0 = \eta|_{D \times Imd}.$$

Then, $(B, D, d, 0, \eta)$ is a braided Γ-crossed module.

We now show that braided Γ-crossed modules are determined by braided strict Γ-graded categorical groups. First, we say that a symmetric factor set (F, θ) on Γ with coefficients in a braided categorical group $\mathbb{G}$ is *regular* if F^σ is a regular symmetric monoidal functor and $\theta^{\sigma,\tau} = id$, for all $\sigma, \tau \in \Gamma$.

Definition 4.5. A braided Γ-graded categorical group $(\mathbb{G}, gr)$ is called *strict* if

i) $\mathrm{Ker}\,\mathbb{G}$ is a braided strict categorical group,

ii) $\mathbb{G}$ induces a regular symmetric factor set (F, θ) on Γ with coefficients in $\mathrm{Ker}\,\mathbb{G}$.

Equivalently, a braided Γ-graded categorical group $(\mathbb{G}, gr)$ is *strict* if it is a Γ-graded extension of a braided strict categorical group by a regular symmetric factor set.

• Constructing the braided strict Γ-graded categorical group $\mathbb{G} = \mathbb{G}_{\mathcal{M}}$ associated to a braided Γ-crossed module $\mathcal{M} = (B, D, d, \vartheta, \eta)$.

Objects of $\mathbb{G}$ are the elements of the group D, a σ-morphism $x \to y$ is a pair (b,σ), where $b \in B, \sigma \in \Gamma$ such that $\sigma x = d(b)y$. The composition of two morphisms is given by

$$(x \overset{(b,\sigma)}{\to} y \overset{(c,\tau)}{\to} z) = (x \xrightarrow{(\tau b+c,\tau\sigma)} z). \tag{5}$$

Since B is a Γ-group, the composition is associative and unitary.

For each morphism (b,σ) in $\mathbb{G}$, we have

$$(b,\sigma)^{-1} = (-\sigma^{-1}b, \sigma^{-1}),$$

and hence $\mathbb{G}$ is a groupoid.

The tensor operation on objects is given by the addition in the group D and, for two morphisms $(x \overset{(b,\sigma)}{\to} y), (x' \overset{(c,\sigma)}{\to} y')$ in $\mathbb{G}$, we define

$$(x \overset{(b,\sigma)}{\to} y) \otimes (x' \xrightarrow{(c,\sigma)} y') = (xx' \xrightarrow{(b+\vartheta_y c,\sigma)} yy'). \tag{6}$$

The functoriality of the tensor operation is implied from the compatibility of the action ϑ with the Γ-action and from the conditions in the definition of a braided Γ-crossed module.

Associativity and unit constraints of the tensor operation are strict. The braiding constraint $\mathbf{c}$ is defined by

$$\mathbf{c}_{x,y} = (\eta(x,y), 1) : xy \to yx.$$

By the relation C_5, $\mathbf{c}_{x,y}$ is actually a morphism in $\mathbb{G}$. Due to the conditions C_3, C_4, the braiding constraint $\mathbf{c}$ is compatible with the associativity constraint $\mathbf{a}$. The naturality of $\mathbf{c}$ follows from the conditions $\Gamma_2, C_1, C_3, C_4, C_6, C_7$.

The Γ-grading $gr : \mathbb{G} \to \Gamma$ is given by

$$(b,\sigma) \mapsto \sigma.$$

The unit graded functor $I : \Gamma \to \mathbb{G}$ is defined by

$$I(* \overset{\sigma}{\to} *) = (1 \xrightarrow{(0,\sigma)} 1).$$

Since $\mathrm{Ob}\mathbb{G} = D$ is a group and $x \otimes y = xy$, every object of $\mathbb{G}$ is invertible, and hence $\operatorname{Ker}\mathbb{G}$ is a braided strict categorical group.

We now show that $\mathbb{G}$ induces a regular symmetric factor set (F,θ) on Γ with coefficients in $\operatorname{Ker}\mathbb{G}$. For any $x \in D, \sigma \in \Gamma$, we set $F^\sigma(x) = \sigma x$, $\Upsilon_x^\sigma = (x \overset{(0,\sigma)}{\to} \sigma x)$. Then, according to the proof of Lemma 2.4, we have $F^\sigma(b,1) = (\sigma b, 1)$ and $\theta^{\sigma,\tau} = id$. From the braided Γ-crossed module structure of $\mathcal{M}$, it follows that F^σ is a regular symmetric monoidal functor on $\operatorname{Ker}\mathbb{G}$.

• Constructing the braided Γ-crossed module *associated* to a braided strict Γ-graded categorical group $\mathbb{G}$.

Set
$$D = \mathrm{Ob}\,\mathbb{G},\ \ B = \{x \xrightarrow{b} 1 \mid x \in D,\ gr(b) = 1\}.$$
The operations in D and B are given by
$$xy = x \otimes y,\ \ b + c = b \otimes c,$$
respectively. Then D becomes a group in which the unity is 1 and the inverse of x is x^{-1} $(x \otimes x^{-1} = 1)$, B is group in which the zero element is the morphism $(1 \xrightarrow{id_1} 1)$ and the inverse of $(x \xrightarrow{b} 1)$ is the morphism $(x^{-1} \xrightarrow{\overline{b}} 1)(b \otimes \overline{b} = id_1)$. Since $\mathbb{G}$ has a regular symmetric factor set (F, θ), D and B are Γ-groups under the actions
$$\sigma x = F^\sigma(x),\ x \in D, \sigma \in \Gamma,$$
$$\sigma b = F^\sigma(b),\ b \in B,$$
respectively. The correspondences $d : B \to D$ and $\vartheta : D \to \mathrm{Aut}\,B$ are, respectively, given by
$$d(x \xrightarrow{b} 1) = x,$$
$$\vartheta_y(x \xrightarrow{b} 1) = (yxy^{-1} \xrightarrow{id_y + b + id_{y^{-1}}} 1).$$
Since B and D are Γ-groups, d and ϑ are Γ-group homomorphisms.

The map $\eta : D \times D \to B$ is defined by
$$\eta(x, y) = \mathbf{c}_{x,y} \otimes id_{x^{-1}} \otimes id_{y^{-1}} : xyx^{-1}y^{-1} \to 1.$$

Now we will classify the category of braided Γ-crossed modules.

Definition 4.6. A *homomorphism* $\mathcal{M} \to \mathcal{M}'$ of braided Γ-crossed modules is a homomorphism (f_1, f_0) of braided crossed modules, where f_1, f_0 are Γ-group homomorphisms.

Remark on notations. Each morphism $x \xrightarrow{(b,\sigma)} y$ in $\mathbb{G}_{\mathcal{M}}$ is written in the form
$$x \xrightarrow{(0,\sigma)} \sigma x \xrightarrow{(b,1)} y,$$
and then each graded symmetric monoidal functor $(F, \widetilde{F}) : \mathbb{G}_{\mathcal{M}} \to \mathbb{G}_{\mathcal{M}'}$ defines a function $f : D^2 \cup (D \times \Gamma) \to B'$ by
$$(f(x, y), 1) = \widetilde{F}_{x,y},\quad (f(x, \sigma), \sigma) = F(x \xrightarrow{(0,\sigma)} \sigma x). \tag{7}$$

Lemma 4.7. *Let $(f_1, f_0) : \mathcal{M} \to \mathcal{M}'$ be a homomorphism of braided Γ-crossed modules. Then there is a graded symmetric monoidal functor $(F, \widetilde{F}) : \mathbb{G}_{\mathcal{M}} \to \mathbb{G}_{\mathcal{M}'}$ such that $F(x) = f_0(x)$, $F(b, 1) = (f_1(b), 1)$, if and only if $f = p^*\varphi$, where $\varphi \in Z^2_{\Gamma,s}(\mathrm{Coker}\,d,\ \mathrm{Ker}\,d')$, and $p : D \to \mathrm{Coker}\,d$ is a canonical projection.*

Proof. Since f_0 is a homomorphism and $Fx = f_0(x)$, $\widetilde{F}_{x,y} : FxFy \to F(xy)$ is a morphism of grade 1 in $\mathbb{G}'$ if and only if $df(x,y) = 1'$, or $f(x,y) \in \operatorname{Ker} d' \subset Z(B')$.

Also, since f_0 is a Γ-homomorphism, $Fx \xrightarrow{(f(x,\sigma),\sigma)} F\sigma x$ is a morphism of grade σ in $\mathbb{G}'$ if and only if $df(x,\sigma) = 1'$, or $f(x,\sigma) \in \operatorname{Ker} d' \subset Z(B')$.

• The condition so that F preserves the composition of two morphisms.

Since f_1 is a group homomorphism, F preserves the composition of two morphisms of grade 1. F preserves the composition of two morphisms in terms of $(0,\sigma)$,

$$(x \xrightarrow{(0,\sigma)} y \xrightarrow{(0,\tau)} z),$$

if and only if

$$\tau f(x,\sigma) + f(\sigma x,\tau) = f(x,\tau\sigma). \tag{8}$$

• The condition so that $\widetilde{F}_{x,y}$ is natural.

- For morphisms of grade 1, we consider the diagram

$$\begin{tikzcd}
F(x)F(y) \arrow[r, "\widetilde{F}_{x,y}"] \arrow[d, "F(b,1)\otimes F(c,1)"'] & F(xy) \arrow[d, "F[(b,1)\otimes(c,1)]"] \\
F(x')F(y') \arrow[r, "\widetilde{F}_{x',y'}"'] & F(x'y')
\end{tikzcd}$$

Since $f_1,\ f_0$ are homomorphisms satisfying the condition H_2,

$$F(b,1) \otimes F(c,1) = F[(b,1) \otimes (c,1)].$$

Then since $f(x,y),\ f(x',y') \in Z(B')$, the above diagram commutes if and only if

$$f(x,y) = f(x',y'),$$

for $x = d(b)x'$, $y = d(c)y'$. Thus, $\widetilde{F}$ defines a function $\varphi : \operatorname{Coker}^2 d \to \operatorname{Ker} d'$,

$$\varphi(r,s) = f(x,y),\ r = px, s = py.$$

- For morphisms in terms of $(0,\sigma)$, we consider a diagram

$$\begin{tikzcd}
F(x)F(y) \arrow[r, "\widetilde{F}_{x,y}"] \arrow[d, "F(0,\sigma)\otimes F(0,\sigma)"'] & F(xy) \arrow[d, "F[(0,\sigma)\otimes(0,\sigma)]"] \\
F(\sigma x)F(\sigma y) \arrow[r, "\widetilde{F}_{\sigma x,\sigma y}"'] & F(\sigma x)(\sigma y) = F\sigma(xy).
\end{tikzcd}$$

According to Proposition 4.2, the above diagram commutes if and only if

$$\sigma f(x,y) + f(xy,\sigma) = f(x,\sigma) + f(y,\tau) + f(\sigma x,\sigma y). \tag{9}$$

• Since the following square commutes

$$\begin{tikzcd}
Fx \arrow[r, "{(f(x,\sigma),\sigma)}"] \arrow[d, "{F(b,1)}"'] & F\sigma x \arrow[d, "{F(\sigma b,1)}"] \\
Fy \arrow[r, "{(f(y,\sigma),\sigma)}"'] & F\sigma y
\end{tikzcd}$$

and f_1 is a Γ-group homomorphism, we have $f(x,\sigma) = f(y,\sigma)$, for $x = d(b)y$. This determines a function $\varphi : \operatorname{Coker} d \times \Gamma \to \operatorname{Ker} d'$,

$$\varphi(r,\sigma) = f(x,\sigma), \;\; r = px.$$

Therefore, we obtain a function

$$\varphi : \operatorname{Coker}^2 d \cup \operatorname{Coker} d \times \Gamma \to \operatorname{Ker} d'.$$

The function φ is normalized in the sense that

$$\varphi(1,r) = \varphi(s,1) = 0 = \varphi(s,1_\Gamma).$$

The first two equalities follow from the property $F(1) = 1'$ and the compatibility of $(F,\widetilde{F})$ with unit constraints. The final equality holds owing to $f(x,1_\Gamma) = 0$ (following from the relation (8)).

By Proposition 4.2, the compatibility of $(F,\widetilde{F})$ with associativity constraints is equivalent to

$$f(y,z) + f(x,yz) = f(x,y) + f(xy,z). \tag{10}$$

The compatibility of $(F,\widetilde{F})$ with the braiding constraints implies

$$f(x,y) + f_1(\eta(x,y)) = \eta'(f_0(x), f_0(y)) + f(y,x).$$

By the fact that $f(x,y) \in \operatorname{Ker} d' \subset Z(B')$ and by the condition H_3, one has

$$f(x,y) = f(y,x). \tag{11}$$

From the relations (8)–(11), it follows that $\varphi \in Z^2_{\Gamma,s}(\operatorname{Coker} d, \operatorname{Ker} d')$. $\square$

We define the category

$${}_\Gamma\mathbf{BrCross}$$

whose objects are braided Γ-crossed modules and whose morphisms are triples

(f_1, f_0, φ), where $(f_1, f_0) : (B \xrightarrow{d} D) \to (B' \xrightarrow{d'} D')$ is a homomorphism of braided Γ-crossed modules, and $\varphi \in Z^2_{\Gamma,s}(\operatorname{Coker} d, \operatorname{Ker} d')$. The composition is given by (4).

Note that a braided strict Γ-graded categorical group $\mathbb{G}$ induces Γ-actions on the group D of objects and on the group B of morphisms of grade 1, we state the following definition.

Definition 4.8. A graded symmetric monoidal functor $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{G}'$ is termed *regular* if

B_1. $F(x) \otimes F(y) = F(x \otimes y)$,

B_2. $F(b) \otimes F(c) = F(b \otimes c)$,

B_3. $\widetilde{F}_{x,y} = \widetilde{F}_{y,x}$,

B_4. $F(\sigma x) = \sigma F(x)$,

B_5. $F(\sigma b) = \sigma F(b)$,

where $x, y \in \mathrm{Ob}\mathbb{G}$, b, c are morphisms of grade 1 in $\mathbb{G}, \sigma \in \Gamma$.

The graded symmetric monoidal functor mentioned in Lemma 4.7 is regular.

Lemma 4.9. *Let $\mathbb{G}$, $\mathbb{G}'$ be corresponding braided strict Γ-graded categorical groups associated to braided Γ-crossed modules $\mathcal{M}$, $\mathcal{M}'$, and let $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{G}'$ be a regular graded symmetric monoidal functor. Then, the triple (f_1, f_0, φ), where*

i) $f_0(x) = F(x)$, $(f_1(b), 1) = F(b, 1)$, $\sigma \in \Gamma, b \in B, x \in D$,

ii) $p^*\varphi = f$, *where f is defined by* (7),

is a morphism in the category ${}_{\mathbf{\Gamma}}\mathbf{BrCross}$.

Proof. Due to the conditions B_1 and B_4, f_0 is a Γ-group homomorphism. By the assumption that F preserves the composition of two morphisms of grade 1 and by the condition B_5, f_1 is a Γ-group homomorphism. Any $b \in B$ can be considered as a morphism $(db \overset{(b,1)}{\to} 1)$ in $\mathbb{G}$, and hence $(f_0(db) \overset{(f_1(b),1)}{\to} 1')$ is a morphism in $\mathbb{G}'$, that is, the relation H_1 holds. The relation H_2 follows from the condition B_2 and the homomorphism property of f_1.

According to the proof of Lemma 4.7, the compatibility of $(F, \widetilde{F})$ with braiding constraints and the condition B_3 lead to the relation H_3. So, (f_1, f_0) is a homomorphism of braided crossed Γ-modules. Thus, by Lemma 4.7, the function f determines a function $\varphi \in Z^2_{\Gamma,s}(\mathrm{Coker}\, d, \mathrm{Ker}\, d')$ such that $f = p^*\varphi$, where $p : D \to \mathrm{Coker}\, d$ is a canonical projection. Therefore, (f_1, f_0, φ) is a morphism in ${}_{\mathbf{\Gamma}}\mathbf{BrCross}$. $\square$

Denote by

$${}_{\mathbf{\Gamma}}\mathbf{BrGr}^*$$

the category of braided strict Γ-graded categorical groups and regular graded symmetric monoidal functors, we obtain the following result which is an extension of Theorem 3.6.

Theorem 4.10 (Classification Theorem)**.** *There exists an equivalence*

$$\begin{array}{rcl} \Phi : {}_{\mathbf{\Gamma}}\mathbf{BrCross} & \to & {}_{\mathbf{\Gamma}}\mathbf{BrGr}^*, \\ B \to D & \mapsto & \mathbb{G}_{B \to D} \\ (f_1, f_0, \varphi) & \mapsto & (F, \widetilde{F}) \end{array}$$

where $F(x) = f_0(x)$, $F(b,1) = (f_1(b),1)$, $F(x \stackrel{(0,\sigma)}{\to} \sigma x) = (\varphi(px,\sigma),\sigma)$, $\widetilde{F}_{x,y} = (\varphi(px,py),1)$, *for* $x \in D, b \in B, \sigma \in \Gamma$.

Proof. Suppose that $\mathbb{G}, \mathbb{G}'$ are braided strict Γ-graded categorical groups associated to braided Γ-crossed modules $B \to D, B' \to D'$, respectively. By Lemma 4.7, the correspondence $(f_1, f_0, \varphi) \mapsto (F, \widetilde{F})$ determines an injection on the homsets

$$\Phi : \mathrm{Hom}_{{}_\Gamma\mathbf{BrCross}}(B \to D, B' \to D') \to \mathrm{Hom}_{{}_\Gamma\mathbf{BrGr}^*}(\mathbb{G}, \mathbb{G}').$$

According to Lemma 4.9, Φ is surjective.

If $\mathbb{G}$ is a braided strict Γ-graded categorical group, and $\mathcal{M}_\mathbb{G}$ is its an associated braided Γ-crossed module, then $\Phi(\mathcal{M}_\mathbb{G}) = \mathbb{G}$ (not only isomorphic). Therefore, Φ is an equivalence. $\square$

Remark 4.11. *In the above theorem, if* $B \to D$ *is a symmetric* Γ*-crossed module, then* $\mathbb{G}_{B\to D}$ *is a symmetric strict* Γ*-graded categorical group. Let* ${}_\Gamma\mathbf{SymCross}$ *denote the full subcategory of the category* ${}_\Gamma\mathbf{BrCross}$ *whose objects are symmetric crossed* Γ*-modules, and let* ${}_\Gamma\mathbf{PiGr}^*$ *denote the full subcategory of the category* ${}_\Gamma\mathbf{BrGr}^*$ *whose objects are symmetric strict* Γ*-graded categorical groups. Then these two subcategories are equivalent and the following diagram commutes*

$$\begin{tikzcd}
{}_\Gamma\mathbf{SymCross} \arrow[r, "\Phi"] \arrow[d, hook, "J"'] & {}_\Gamma\mathbf{PiGr}^* \arrow[d, hook, "J^*"] \\
{}_\Gamma\mathbf{BrCross} \arrow[r, "\Phi"'] & {}_\Gamma\mathbf{BrGr}^*,
\end{tikzcd}$$

where J, J^* *are full embedding functors.*

Remark 4.12. *When* $\Gamma = 1$ *is a trivial group, then the categories* ${}_\Gamma\mathbf{BrCross}$ *and* ${}_\Gamma\mathbf{BrGr}^*$ *are the categories* $\mathbf{BrCross}$ *and* $\mathbf{BrGr}^*$*, respectively. Therefore, we obtain Theorem 3.6.*

5 Classification of Γ-module extensions of the type of an abelian Γ-crossed module

In this section, we present the theory of Γ-module extension of the type of an abelian Γ-crossed modules, which is analogous to the theory of group extension of the type of a crossed module [19, 10, 3].

In [6], if $d : B \to D$ is a homomorphism of abelian groups and D acts trivially on B, then $(B, D, d, 0)$ is called an *abelian crossed module.* Let us note that any abelian crossed module is defined by a strict Picard category,

that is, a symmetric categorical group in which $\mathbf{a} = id$, $\mathbf{c} = id$, $\mathbf{l} = id = \mathbf{r}$ and for each object x, there is an object y such that $x \otimes y = 1$).

By an *abelian* Γ-crossed module, we shall mean a braided Γ-crossed module
$(B, D, d, \vartheta, \eta)$ that $\vartheta = 0$, $\eta = 0$. Then d is a homomorphism of Γ-modules.

According to the construction in Section 4, each abelian Γ-crossed module $\mathcal{M} = (B, D, d)$ defines a Γ-graded category $\mathbb{G}_{\mathcal{M}}$ whose $\operatorname{Ker}\mathbb{G}$ is a strict Picard category. In this case, we say that $\mathbb{G}_{\mathcal{M}}$ is a strict Γ-graded Picard category. A *homomorphism* $(f_1, f_0) : (B, D, d) \to (B', D', d')$ of abelian Γ-crossed modules consists of Γ-module homomorphisms $f_1 : B \to B'$ and $f_0 : D \to D'$ such that

$$f_0 d = d' f_1.$$

Note that in this section, since B and D are abelian groups, we write $+$ for the operations on B, D.

Definition 5.1. Let $\mathcal{M} = (B, D, d)$ be an abelian Γ-crossed module, and let Q be a Γ-module. A Γ-module extension of B by Q *of type* $\mathcal{M}$*,* denoted by $\mathcal{E}_{d,Q}$, is a short exact sequence of Γ-module homomorphisms,

$$\mathcal{E} : 0 \to B \xrightarrow{j} E \xrightarrow{p} Q \to 0,$$

and a homomorphism of abelian Γ-crossed modules $(id, \varepsilon) : (B \to E) \to (B \to D)$.

Two extensions $\mathcal{E}_{d,Q}$ and $\mathcal{E}'_{d,Q}$ are said to be *equivalent* if the following diagram commutes

$$\begin{tikzcd}
\mathcal{E} : 0 \arrow[r] & B \arrow[r, "j"] \arrow[d, equal] & E \arrow[r, "p"] \arrow[d, "\alpha"] & Q \arrow[r] \arrow[d, equal] & 0, & E \arrow[r, "\varepsilon"] & D \\
\mathcal{E} : 0 \arrow[r] & B \arrow[r, "j'"] & E' \arrow[r, "p'"] & Q \arrow[r] & 0, & E \arrow[r, "\varepsilon'"] & D
\end{tikzcd}$$

and $\varepsilon' \alpha = \varepsilon$. Obviously, α is an isomorphism of Γ-modules.

Each extension $\mathcal{E}_{d,Q}$ induces a Γ-module homomorphism $\psi : Q \to \operatorname{Coker} d$ such that $\psi p = q\varepsilon$, where $q : D \to \operatorname{Coker} d$ is a canonical projection. Moreover, ψ is dependent only on the equivalence class of the extension $\mathcal{E}_{d,Q}$, and then we say that $\mathcal{E}_{d,Q}$ *induces* ψ. The set of equivalence classes of extensions $\mathcal{E}_{d,Q}$ inducing $\psi : Q \to \operatorname{Coker} d$ is denoted by

$$\operatorname{Ext}^{\mathcal{M}}_{\mathbb{Z}\Gamma}(Q, B, \psi).$$

Now, in order to study this set we apply the obstruction theory for graded symmetric monoidal functors between strict Γ-graded Picard categories $\operatorname{Dis}_{\Gamma,s} Q$ and $\mathbb{G}_{B \to D}$, where the *discrete* Γ-graded Picard category $\operatorname{Dis}_{\Gamma,s} Q$ is defined by (see Subsection 2.2)

$$\operatorname{Dis}_{\Gamma,s} Q = \int_{\Gamma} (Q, 0, 0).$$

It is just the strict Γ-graded Picard category associated to the abelian Γ-crossed module $(0, Q, 0)$ (see Section 4).

Lemma 5.2. *Let $\mathcal{M} = (B, D, d)$ be an abelian Γ-crossed module, Q be a Γ-module and $\psi : Q \to \operatorname{Coker} d$ be a Γ-module homomorphism. Then for each graded symmetric monoidal functor $(F, \widetilde{F}) : \operatorname{Dis}_{\Gamma,s} Q \to \mathbb{G}_{\mathcal{M}}$ which satisfies $F(0) = 0$ and induces the pair of Γ-module homomorphisms $(\psi, 0) : (Q, 0) \to (\operatorname{Coker} d, \operatorname{Ker} d)$, there exists an extension $\mathcal{E}_{d,Q}$ inducing ψ.*

Such an extension $\mathcal{E}_{d,Q}$ is called *associated* to the graded symmetric monoidal functor $(F, \widetilde{F})$.

Proof. Suppose that $(F, \widetilde{F}) : \operatorname{Dis}_{\Gamma,s} Q \to \mathbb{G}_{\mathcal{M}}$ is a graded symmetric monoidal functor. Then $(F, \widetilde{F})$ determines a function $f : Q^2 \cup (Q \times \Gamma) \to B$ by (7),

$$(f(u, v), 1) = \widetilde{F}_{u,v}, \quad (f(u, \sigma), \sigma) = F(u \xrightarrow{(0,\sigma)} \sigma u).$$

The function f is "normalized" in the sense that

$$f(u, 1_\Gamma) = 0, f(u, 0) = 0 = f(0, v).$$

Since F preserves the identity morphism, one has the first equality. The later equalities follow from the assumption $F(0) = 0$ and the compatibility of $(F, \widetilde{F})$ with unit constraints. It follows from the definition of a morphism in $\mathbb{G}$ that

$$\sigma F(u) = df(u, \sigma) + F(\sigma u), \tag{12}$$

$$F(u) + F(v) = df(u, v) + F(u + v). \tag{13}$$

The function f defined as above is just a 2-cocycle in $Z^2_{\Gamma,s}(Q, B)$.

From the 2-cocycle f, we construct an exact sequence of Γ-modules

$$\mathcal{E}_F : \quad 0 \to B \overset{j_0}{\to} E_0 \overset{p_0}{\to} Q \to 0,$$

where E_0 is the crossed product extension $B \times_f Q$ and $j_0(b) = (b, 1)$, $p_0(b, u) = u$, for $b \in B, u \in Q$. The Γ-module structure of E_0 is given by

$$(b, u) + (c, v) = (b + c + f(u, v), u + v),$$

$$\sigma(b, u) = (\sigma b + f(u, \sigma), \sigma u).$$

Now we determine Γ-module homomorphism $\varepsilon : E_0 \to D$. By the assumption, $(F, \widetilde{F})$ induces a Γ-module homomorphism $\psi : Q \to \operatorname{Coker} d$ by $\psi(u) = [Fu] \in \operatorname{Coker} d$. Thus, the element Fu is a representative of $\operatorname{Coker} d$ in D. Then for $(b, u) \in E_0$, we set

$$\varepsilon(b, u) = db + Fu. \tag{14}$$

Therefore, ε is a Γ-module homomorphism thanks to the relations (12) and (13). It is easy to see that $\varepsilon \circ j_0 = d$. Further, this extension induces Γ-module homomorphism $\psi : Q \to \operatorname{Coker} d$, since

$$\varepsilon(b, u) = q(db + Fu) = q(Fu) = \psi(u) = \psi p_0(b, u),$$

for all $u \in Q$. □

Theorem 5.3 (Schreier Theory for Γ-module extensions of the type of an abelian Γ-crossed module)**.** *Let $\mathcal{M} = (B, D, d)$ be an abelian Γ-crossed module, and let $\psi : Q \to \operatorname{Coker} d$ be a Γ-module homomorphism. There exists a bijection*

$$\Omega : \operatorname{Hom}_{(\psi,0)}[\operatorname{Dis}_{\Gamma,s} Q, \mathbb{G}_{\mathcal{M}}] \to \operatorname{Ext}^{\mathcal{M}}_{\mathbb{Z}\Gamma}(Q, B, \psi).$$

Proof. Step 1: Graded symmetric monoidal functors $(F, \widetilde{F})$, $(F', \widetilde{F}')$ are homotopic if and only if the corresponding associated extensions $\mathcal{E}_{d,Q}, \mathcal{E}'_{d,Q}$ are equivalent.

Suppose that $F, F' : \operatorname{Dis}_{\Gamma,s} Q \to \mathbb{G}_{\mathcal{M}}$ are homotopic by a homotopy $\alpha : F \to F'$. Then, there is a function $g : Q \to B$ such that $\alpha_u = (g(u), 1)$, that is,

$$F(u) = dg(u) + F'(u). \tag{15}$$

The naturality and the coherence condition (3) of the homotopy α lead to $g(0) = 0$ and

$$f(u, \sigma) + g(\sigma u) = \sigma g(u) + f'(u, \sigma), \tag{16}$$

$$f(u, v) + g(u + v) = g(u) + g(v) + f'(u, v). \tag{17}$$

According to Lemma 5.2, there exist the extensions $\mathcal{E}_{d,Q}$ and $\mathcal{E}'_{d,Q}$ associated to F and F', respectively. Then, thanks to the relations (16) and (17), the map

$$\alpha^* : E_F \to E_{F'}, \quad (b, u) \mapsto (b + g(u), u)$$

is a homomorphism of Γ-modules. Further, α^* is an isomorphism. The equality $\varepsilon'\alpha^* = \varepsilon$ is implied from the relations (14) and (15):

$$\begin{aligned}\varepsilon'\alpha^*(b, u) &= \varepsilon'(b + g(u), u) = d(b + g(u)) + F'u \\ &= d(b) + d(g(u)) + F'u = d(b) + Fu = \varepsilon(b, u).\end{aligned}$$

Therefore, two extensions $\mathcal{E}_{d,Q}$ and $\mathcal{E}'_{d,Q}$ are equivalent.

Now, suppose that $\mathcal{E}_{d,Q}$ and $\mathcal{E}'_{d,Q}$ are two extensions associated to $(F, \widetilde{F})$ and $(F', \widetilde{F}')$, respectively. If $\alpha^* : E_F \to E_{F'}$ is an equivalence of these extensions, then it is straightforward to see that

$$\alpha^*(b, u) = (b + g(u), u),$$

where $g : Q \to B$ is a function with $g(0) = 0$. By retracing our steps, $\alpha_u = (g(u), 1)$ is a homotopy between $(F, \widetilde{F})$ and $(F', \widetilde{F}')$.

Step 2: Ω is surjective.

Assume that $\mathcal{E} = \mathcal{E}_{d,Q}$ is an extension of type $\mathcal{M}$. We prove that $\mathcal{E}$ defines a graded symmetric monoidal functor $(F, \widetilde{F}) : \mathrm{Dis}_{\Gamma,s}\, Q \to \mathbb{G}_{\mathcal{M}}$. For any $u \in Q$, choose a representative $e_u \in E$ such that $p(e_u) = u,\ e_0 = 0$. Each element of E can be represented uniquely as $b + e_u$ for $b \in B, u \in Q$. The representatives $\{e_u\}$ induce a normalized function $f : Q^2 \cup (Q \times \Gamma) \to B$ by

$$e_u + e_v = f(u, v) + e_{u+v}, \tag{18}$$

$$\sigma e_u = f(u, \sigma) + e_{\sigma u}. \tag{19}$$

Now, we construct a graded symmetric monoidal functor $(F, \widetilde{F})$: $\mathrm{Dis}_{\Gamma,s}\, Q \to \mathbb{G}_{\mathcal{M}}$ as follows. Since $\psi(u) = \psi p(e_u) = q\varepsilon(e_u)$, $\varepsilon(e_u)$ is a representative of $\psi(u)$ in D. Thus, we set

$$F(u) = \varepsilon(e_u),\ \ F(u \xrightarrow{\sigma} \sigma u) = (f(u, \sigma), \sigma),\ \ \widetilde{F}_{u,v} = (f(u, v), 1).$$

The relations (18) and (19) show that $F(\sigma)$ and $\widetilde{F}_{u,v}$ are morphisms in $\mathbb{G}$, respectively. The associativity and commutativity laws and the Γ-group property of B show that $f \in Z^2_{\Gamma,s}(Q, B)$, and hence $(F, \widetilde{F})$ is a graded symmetric monoidal functor of type $(\psi, 0)$.

Now, let $\mathcal{E}_F$ be an extension associated to $(F, \widetilde{F})$, then $\mathcal{E}_F \cong \mathcal{E}$ by α : $(b, u) \mapsto b + e_u$. □

Let $\mathbb{G}$ be a Γ-graded Picard category associated to an abelian Γ-crossed module $B \xrightarrow{d} D$. Since $\pi_0\mathbb{G} = \mathrm{Coker}\, d$ and $\pi_1\mathbb{G} = \mathrm{Ker}\, d$, it follows from Subsection 2.2 that the reduced Γ-graded Picard category $\mathbb{G}(h)$ of $\mathbb{G}$ is of the form

$$\mathbb{G}(h) = \int_\Gamma (\mathrm{Coker}\, d, \mathrm{Ker}\, d, h),\ \ h \in Z^3_{\Gamma,s}(\mathrm{Coker}\, d, \mathrm{Ker}\, d).$$

Then Γ-module homomorphism $\psi : Q \to \mathrm{Coker}\, d$ induces an obstruction

$$\psi^* h \in Z^3_{\Gamma,s}(Q, \mathrm{Ker}\, d).$$

We now use this notion of obstruction to state and prove the following theorem.

Theorem 5.4. *Let $\mathcal{M} = (B, D, d)$ be an abelian Γ-crossed module, and let $\psi : Q \to \mathrm{Coker}\, d$ be a homomorphism of Γ-modules. Then, the vanishing of $\overline{\psi^* h}$ in $H^3_{\Gamma,s}(Q, \mathrm{Ker}\, d)$ is necessary and sufficient for there to exist an extension $\mathcal{E}_{d,Q}$ of type $\mathcal{M}$ inducing ψ. Further, if $\overline{\psi^* h}$ vanishes, then the set of equivalence classes of such extensions is bijective with $H^2_{\Gamma,s}(Q, \mathrm{Ker}\, d)$.*

Proof. By the assumption $\overline{\psi^* h} = 0$, it follows from Proposition 2.2 that there is a graded symmetric monoidal functor $(\Psi, \widetilde{\Psi}) : \mathrm{Dis}_{\Gamma,s}\, Q \to \mathbb{G}(h)$. Then the composition of $(\Psi, \widetilde{\Psi})$ and the canonical graded symmetric monoidal functor $(H, \widetilde{H}) : \mathbb{G}(h) \to \mathbb{G}$ is a graded symmetric monoidal functor $(F, \widetilde{F}) : \mathrm{Dis}_{\Gamma,s}\, Q \to \mathbb{G}$, and hence by Lemma 5.2, we obtain an associated extension $\mathcal{E}_{d,Q}$.

Conversely, suppose that

$$\mathcal{E} : 0 \to B \xrightarrow{j} E \xrightarrow{p} Q \to 0$$

is a Γ-module extension of type $\mathcal{M}$ inducing ψ. Let $\mathbb{G}'$ be a strict Γ-graded Picard category associated to the abelian Γ-crossed module (B, E, j). Then, according to Proposition 4.7, there is a graded symmetric monoidal functor $F : \mathbb{G}' \to \mathbb{G}$. Since the reduced Γ-graded Picard category of $\mathbb{G}'$ is $\mathrm{Dis}_{\Gamma,s}\, Q$, it follows from Proposition 2.1 that F induces a graded symmetric monoidal functor of type $(\psi, 0)$ from $(Q, 0, 0)$ to $(\mathrm{Coker}\, d, \mathrm{Ker}\, d, h)$. Now, thanks to Proposition 2.2, the obstruction of the pair $(\psi, 0)$ vanishes in $H^3_{\Gamma,s}(Q, \mathrm{Ker}\, d)$, i.e., $\overline{\psi^* h} = 0$.

The final assertion of Theorem 5.4 is obtained from Theorem 5.3. First, there is a natural bijection

$$\mathrm{Hom}[\mathrm{Dis}_{\Gamma,s}\, Q, \mathbb{G}] \leftrightarrow \mathrm{Hom}\, \mathrm{Dis}_{\Gamma,s}\, Q, \mathbb{G}(h)].$$

Then, since $\pi_0(\mathrm{Dis}_{\Gamma,s}\, Q) = Q, \pi_1 \mathbb{G}(h) = \mathrm{Ker}\, d$, the bijection

$$\mathrm{Ext}^{\mathcal{M}}_{\mathbb{Z}\Gamma}(Q, B, \psi) \leftrightarrow H^2_{\Gamma,s}(Q, \mathrm{Ker}\, d)$$

follows from Theorem 5.3 and Proposition 2.2. □

We now consider the special case when $\mathcal{M} = (B, \mathrm{Aut}\, B, 0)$ is an abelian Γ-crossed module. Then, each Γ-module extension of type $\mathcal{M}$ inducing $\psi : Q \to \mathrm{Aut}\, B$ is just an extension of Γ-modules,

$$0 \to B \to E \to Q \to 0,$$

inducing ψ. Thus, Theorem 5.4 leads to the following consequence.

Corollary 5.5 ([8], Theorem 2.4)**.** *Let B, Q be Γ-modules, and let $\psi : Q \to \mathrm{Aut}\, B$ be a Γ-module homomorphism. Then, there is an obstruction class $\overline{k} \in H^3_{\Gamma,s}(Q, B)$ whose vanishing is necessary and sufficient for there to exist a Γ-module extension of B by Q inducing ψ. Further, if $\overline{k}$ vanishes, then there exists a bijection*

$$\mathrm{Ext}_{\mathbb{Z}\Gamma}(Q, B, \psi) \leftrightarrow H^2_{\Gamma,s}(Q, B).$$

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